WNE Linear Algebra
Resit Exam
Series B

24 Feburary 2024

Please use separate sheets for different problems. Please use single sheet for all questions. Give reasons to your answers. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each problem is worth 10 marks. Each question is worth 4 marks.

Problems

Problem 1.
Let $V = \text{lin}((1,1,-1,-4),(1,2,-1,-7),(-3,2,3,-3))$ be a subspace of $\mathbb{R}^4$.

a) find a basis $\mathcal{A}$ of the subspace V and the dimension of V,
b) find coordinates of vector $v = (3,1,-3,-6)$ relative to $\mathcal{A}$.

Problem 2.
Let $V \subset \mathbb{R}^4$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} x_1 + x_2 + x_4 = 0 \\ 2x_1 + 3x_2 + 3x_3 + 4x_4 = 0 \end{cases}$$

a) find a basis $\mathcal{A}$ and the dimension of the subspace V,
b) complete basis $\mathcal{A}$ to a basis of $\mathbb{R}^4$.

Problem 3.
Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 4 & 2 \\ 0 & t & 1 \end{bmatrix}.$$

a) find all $t \in \mathbb{R}$ such that matrix $(A^{\mathsf{T}})^8$ is invertible,
b) for $t = 1$ find A^{-1}.

Problem 4.
Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an endomorphism given by the formula

$$\varphi((x_1,x_2,x_3)) = (x_1 + 2x_2, 2x_1 + 4x_2, -x_3).$$

a) find eigenvalues and bases of the corresponding eigenspaces of φ,
b) is endomorphism φ diagonalizable? is matrix $M(\varphi)_{st}^{st}$ positive semidefinite? Justify your answers.

Problem 5.
Let

$$V = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 + 3x_2 + 2x_3 = 0\},$$

be a subspace of $\mathbb{R}^3$.

a) find an orthonormal basis of V and an orthonormal basis of $V^{\perp}$,
b) find the formula of the orthogonal projection onto $V^{\perp}$.

Problem 6.
Consider the following linear programming problem $2x_2 + x_3 + x_4 \to \min$ in the standard form with constraints

$$\begin{cases} x_1 \qquad\quad + x_3 \qquad\qquad\quad = 5 \\ \qquad x_2 \qquad\quad + x_4 \qquad = 1 \\ x_1 + 5x_2 \qquad\qquad\quad + x_5 = 10 \end{cases} \quad \text{and } x_i \geqslant 0 \text{ for } i = 1, \dots, 5.$$

a) which of the sets $\mathcal{B}_1 = \{3, 4, 5\}$, $\mathcal{B}_2 = \{1, 3, 4\}$ is basic feasible? write the corresponding basic solutions for both sets,
b) solve the linear programming problem using simplex method. Start from the basic feasible set taken from part a).

Questions

Question 1.
Let $A, B \in M(2 \times 2; \mathbb{R})$ be two symmetric matrices, that is $A^\intercal = A, B^\intercal = B$. If A is positive definite and B is negative definite, does it follow that $A + B$ is indefinite?

Solution 1.
No, it does not.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Question 2.
Let $A \in M(2 \times 2; \mathbb{R})$ be a matrix. Does it follow that

$$A = -A^\intercal \quad \Longleftrightarrow \quad (A + A^\intercal)^2 = 0.$$

Solution 2.
Yes, it does.
$(\Longrightarrow)$ trivial
$(\Longleftarrow)$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$(A + A^\intercal)^2 = \begin{bmatrix} (c+b)^2 + 4a^2 & 2(c+b)\,d + 2a\,(c+b) \\ 2(c+b)\,d + 2a\,(c+b) & 4d^2 + (c+b)^2 \end{bmatrix}.$$

In particular,

$$(c+b)^2 + 4a^2 = 4d^2 + (c+b)^2 = 0,$$

therefore $a = d = 0$ and $c = -b$. Note that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^2 = 0 \quad \text{but} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0.$$

Question 3.
If $a + b + c = 0$ where $a, b, c \in \mathbb{R}$, does it follow that

$$\det \begin{bmatrix} a^2 & ab & ac \\ b^2 & bc & ab \\ c^2 & ac & bc \end{bmatrix} = 0.$$

Solution 3.
Yes, it does.

$$\det \begin{bmatrix} a^2 & ab & ac \\ b^2 & bc & ab \\ c^2 & ac & bc \end{bmatrix} = abc \det \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} = \det \begin{bmatrix} a+b+c & a+b+c & a+b+c \\ b & a & c \\ c & b & a \end{bmatrix} = 0.$$

Question 4.
Let $A, B \in M(2 \times 2; \mathbb{R})$. Assume that A and B are simultaneously diagonalizable, that is there exists an invertible matrix $C \in M(2 \times 2; \mathbb{R})$ such that matrices $C^{-1}AC$ and $C^{-1}BC$ are diagonal. Does it follow that $AB = BA$?

Solution 4.
Yes, it does. Since diagonal matrices commute

$$C^{-1}ACC^{-1}BC = C^{-1}BCC^{-1}AC \quad \Longrightarrow \quad AB = BA.$$

Question 5.
Let $A, B \in M(2 \times 2; \mathbb{R})$ be two matrices. If A and B are not invertible, does it follow that $A + B$ is not invertible?

Solution 5.
No, it does not.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$